

Exact closed form analytical solutions for vibrating cavities

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For one-dimensional vibrating cavity systems appearing in the standard illustration of the dynamical Casimir effect, we propose an approach to the construction of exact closed-form solutions. As new results, we obtain solutions that are given for arbitrary frequencies, amplitudes and time regions. In a broad range of parameters, a vibrating cavity model exhibits the general property of exponential instability. Marginal behavior of the system manifests in a power-like growth of radiated energy.

PACS numbers: 42.50.Lc, 03.70.+k, 11.10.-z

I. INTRODUCTION

The dynamical variety of the Casimir effect has been studied in numerous papers in the last two decades [1, 2]. The most favorable model for theoretical considerations refers to the original Casimir's setup [3] of a cavity composed of two perfectly conducting parallel plates. The dynamical modification means that the distance between plates changes with time. Then, we come across new fascinating phenomena in the context of quantum field theory. The most impressive manifestation of unusual non-classical properties of the theory of quantized fields is the effect of particle production "from nothing". It is referred as dynamical Casimir effect (DCE) or motion induced radiation (MIR). Most of theoretical papers explore so-called vibrating cavities, where oscillations of cavity walls are periodic in time. Attention of their authors is attracted by the instability of quantum fluctuations due to the parametric resonance. They hope the resonance enhancement of particle production could lead to macroscopic effects that would be experimentally detected. Nowadays, the resonant instability of the vacuum that is followed by an explosive particle production in a vibrating cavity is believed to be accessible for measurements [4, 5, 6, 7, 8]. However, we are still awaiting for confirmation of successful experimental projects.

In this paper, we consider the behavior of the electromagnetic field in a one-dimensional vibrating cavity. This problem was studied initially by Moore [9]. A cavity with one stationary wall and one moving wall with some prescribed trajectory had been elaborated there. Moore did not solve any particularly interesting cavity model, but he made a thorough study of a general theory of solving such models. His basic approach was generally used and developed in next years [10]-[72]. From theoretical point of view, this simplified model deals with several hard and important problems. We mean the task of solving a wave equation with time-dependent boundary conditions, the difficulties with analytical description of physical systems under the parametric resonance conditions, the problem of quantization of fields in limited regions with moving boundaries, the squeezing of quantum states or the problems of quantum entanglement and decoherence. One-dimensional vibrating cavities provide the simplest theoretical laboratories to study these issues in quantum field theory. The methods that are worked out there can be essentially adapted for more complex models. Some results and ideas are endorsed as well. There are advanced approaches to include non-perfect [55, 56] or partly transmitting [72] cavity walls, finite temperature effects [32, 46, 47] or proceed to three-dimensional case [25, 31, 39, 40, 49, 51, 52, 62].

In the case of one-dimensional cavities and "scalar electrodynamics", main achievements of numerous investigations were obtained either in the framework of the effective Hamiltonian approach [1, 23, 35] or using various numerical approaches [24, 48, 54, 66, 67, 69, 70]. Analytical solutions obtained through the perturbation methods with effective Hamiltonians hold only for small amplitudes [28, 29, 37] or for particular time regimes, either short time [30] or long time [20] limits. In many investigations, a frequency of cavity vibrations is assumed to match a resonance frequency. It is well known from classical mechanics [73] that parametric resonance occurs also for frequencies that are not finely tuned provided that respective amplitudes of oscillations are sufficiently large. Off resonant behavior of vibrating cavities is usually studied in the limit of small detuning from resonance frequencies [38, 58].

Our aim is to gather exact analytical and global solutions for vibrating cavities. In fact, there are few known solutions that can be described by closed form expressions. The first closed form exact solution to describe a cavity vibrating at its resonance frequency was presented by Law [21]. Law's solution corresponds to a cavity that oscillates

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basically sinusoidally for small amplitudes. The frequency of oscillations is twice the lowest eigenfrequency of the cavity (so-called "principal resonance" [1]). Law found travelling wave packets in the energy density of the field. He noted "sub-Casimir" quantum fluctuations far away from the wave packets. Next, Wu et al. [41] presented a family of exact analytical solutions for all resonance frequencies. In the particular case of the second resonance frequency, their solution is matching Law's solution. They described emerging of wave packets in the energy density, indicated sub-Casimir fluctuations and emphasized the absence of wave packets in the first resonant channel with the fundamental cavity eigenfrequency ("semi-resonance" [1]). One can be puzzled that for any one of known exact solutions, there appears a power-like resonance instability. Faithfully, the total energy of the field increases quadratically with time there. On the other hand, it is well recognized from other cavity models that the total radiated energy typically grows exponentially with time [24, 27, 37, 43, 44, 48]. For instance, one can refer to the asymptotic formulas found by Dodonov et al. [1] for cavities that undergo harmonic oscillations. It was generally argued [24, 27] that an exponential resonant instability is typical for vibrating cavities, while a power-like behavior constitutes a critical boundary between stability and instability regions defined by domains of parameters [63]. In this paper, we find solutions that reveal exponential instability generally and exhibit a power-like law as a marginal effect. Moreover, all previously known exact solutions applied only to resonance frequencies. Here, we provide exact solutions that are adjustable for all frequencies. This paper presents a rich class of exact and closed-form solutions, in addition all formerly presented solutions [1, 21, 41] are captured here as particular cases and examined in a more comprehensive way.

Our paper is organized as follows. In Section II, we present our way of representing solutions to describe the quantum dynamics in a vibrating cavity. It relies on $SL(2, R)$ symmetry of the algebraic structure that exists for the quantized scalar field in a static cavity [50, 64, 74]. Actually, we abandon Moore's function that is awkward in use. We put forward another object, called a fundamental map by us, that has remarkable analytical properties. These properties are collected in Section III, together with appropriate mathematical formulae for primary physical functions, namely the vacuum expectation of the energy density of the field inside a cavity and the total radiated energy. Finally, in Section IV we present a collection of exact closed-form solutions. The results are summarized in Section V.

II. REPRESENTATION OF SOLUTIONS TO DESCRIBE THE QUANTUM DYNAMICS IN A VIBRATING CAVITY

In the standard physical setup, we have an electromagnetic resonator of length L composed of two perfectly reflecting walls. Initially, the cavity is static. Then, it undergoes vibrations with a constant frequency ω . In literature, it is frequently assumed that the cavity length L is related with the period of oscillations $T = 2\pi/\omega$. In this paper, we will keep the parameters L and T independent. The parameters provide the characteristic physical length scales. The static cavity length L defines the magnitude of Casimir interactions. In particular, it specifies the scale of quantum fluctuations leading to the production of particles. The period T is the scale of parametric excitations of the system caused by some external force. It is very useful in numerical computations to put $T = \pi$. The parametric resonance is expected when L and T are of the same order. Eventually, it depends also on an amplitude of vibrations. We are willing to yield a phase diagram (Arnold's diagram [73]) that exhibits stability and instability regions.

The derivation of the simplest mathematical model leads to the quantization of free scalar field $A(x, t)$ with Dirichlet boundary conditions imposed at the boundary walls $x = 0$ and $x = L(t)$. The trajectory of the oscillating wall is periodic: $L(t + T) = L(t)$. It is important to assume that $L(t) > 0$ (the cavity never collapses) and $|\dot{L}(t)| \leq v_{max} < 1$ (the wall velocity does not come near the speed of light). Moreover, we impose $L(t) = L$ for $t < 0$ (the cavity is static in the past, this condition is important for the quantization). The construction of the basic set of solutions for this problem was given in [9]:

$$A_N(t, x) = \frac{i}{\sqrt{4\pi N}} [\exp(-i\omega_N R(t + x)) - \exp(-i\omega_N R(t - x))] . \quad (1)$$

The cavity eigenfrequencies $\omega_N = N\pi/L$ are called resonance frequencies. We expect that the parametric resonance occurs at these frequencies for any amplitudes. However, the instability of the system may appear also for other frequencies provided that the amplitude of oscillations is sufficiently large. Usually, it is a hard task to get the picture of the asymptotic behavior of the system for any frequencies and amplitudes. Our knowledge of the system comes to us through the Moore's function R given by the following equation:

$$R(t + L(t)) - R(t - L(t)) = 2L . \quad (2)$$

Usually, Moore's function R is defined as a dimensionless function (phase function). In this paper, we will prefer to define this function in dimensions of length. There is no general theory of solving Eq.(2). Before we present a big set

of exact solutions of the above problem, it is worth to recall some useful symmetry of the static cavity system [50, 64]. In the static region for $t < 0$, the quantized theory is invariant under the conformal transformations:

$$t \pm x \rightarrow R_{min}(t \pm x) , \quad (3)$$

with the functions R_{min} defined by:

$$R_{min}(\tau) = \frac{2}{\omega_1} \arctan \left(\sigma \left(\tan \frac{\omega_1 \tau}{2} \right) \right) , \quad (4)$$

where $\sigma(\tau) = (A\tau + B)/(C\tau + D)$ is any homography and ω_1 is the lowest resonance frequency. Subsequent branches of multivalued function \arctan should be always chosen and linked together in such a way that a resulted function R_{min} is continuous. It is described here a well-known $SL(2, R)$ symmetry of free scalar fields quantized on a strip [74]. Surprisingly, this symmetry is rarely exploited in numerous papers on physical models of the quantum field in a one-dimensional cavity. In particular, the symmetry helps to solve the puzzling problem why there is no resonant behavior of the system for the fundamental resonance frequency $\omega_1 = \pi/L$.

In this paper, we will be searching for exact solutions of Eq.(2) in the following form:

$$R(\tau) = \frac{2}{\omega} \arctan \left(\Delta_{n(\tau)} \left(\tan \frac{\omega \tau}{2} \right) \right) + shift . \quad (5)$$

In order to obtain closed-form solutions, we assume the range of \arctan to be $[-\pi/2, \pi/2]$ (principal branch) and appropriate shifts will be explicitly specified throughout. For instance, the linear Moore's function, which describes a static cavity, should be represented as:

$$R_{static}(\tau) = \tau - \frac{4\pi}{\omega} = \frac{2}{\omega} \arctan \left(\tan \frac{\omega \tau}{2} \right) + \left\lfloor \frac{\omega \tau}{2\pi} - \frac{3}{2} \right\rfloor \frac{2\pi}{\omega} , \quad (6)$$

where we have used the standard notation for the floor function. The construction of the representation Eq.(5) is tied up with the well-known idea from classical mechanics [73]. To explore the dynamics of periodic systems with parametric resonance, it is a handy way to deal with mappings for single periods. Here, we need a set of maps Δ_n numerated by the number n . Fortunately, the maps are not independent. We prove that it is enough to specify only the first map Δ_1 . Henceforth, a function $\Delta_1(v)$ is going to be called a fundamental map throughout this paper. This map defines the auxiliary function f :

$$f(\tau) = \frac{2}{\omega} \arctan \left(\Delta_1 \left(\tan \frac{\omega \tau}{2} \right) \right) + shift , \quad (7)$$

which is a solution of a simpler problem than Eq.(2) (see equations for billiard functions in [63]):

$$f(t + L(t)) = t - L(t) . \quad (8)$$

Since the cavity is static in the past, we have always that $f(\tau) = \tau - 2L$ for $\tau < L$. The subject is also simplified due to the fact that the auxiliary function f fulfils the periodicity condition:

$$f(\tau + T) = f(\tau) + T . \quad (9)$$

In general, the Moore's function $R(\tau)$ is not subject to any periodic conditions. The reason lies in the lack of periodicity of the index $n(\tau)$, that assigns a map to a particular point.

It is straightforward to prove that a fundamental map $\Delta_1(v)$ designates unambiguously a Moore's function $R(\tau)$. The solution of Moore's equation (2) can be build according to the formula:

$$R(\tau) = f^{\circ n(\tau)}(\tau) + 2L[n(\tau) - 1] . \quad (10)$$

Looking at the representation Eq.(5), one can check easily: $\Delta_n = (\Delta_1)^{\circ n}$. Throughout this paper, we use $(\Delta_1)^{\circ n}$ to note n -fold composition $\Delta_1 \circ \Delta_1 \circ \dots \circ \Delta_1$. It remains only to describe the step function $n(\tau)$ that appears in Eq.(5) and Eq.(10). As the function $f(\tau)$ is increasing, the region for $\tau \geq L$ can be covered by intervals $[L_{n-1}, L_n)$, where $L_n \equiv (f^{-1})^{\circ n}(L)$. The map number $n(\tau)$ equals n if the point τ lies inside $[L_{n-1}, L_n)$. Map markers L_n will be called milestones throughout this papers. If $\tau \in [L_{n-1}, L_n)$, then $f(\tau) \in [L_{n-2}, L_{n-1})$. Thus, it is easy to find the following recurrence relation, which is also very convenient for numerical purposes:

$$n(\tau) = \begin{cases} 0 & \tau < L \\ 1 + n(f(\tau)) & \tau \geq L \end{cases} \quad (11)$$

In order to provide a glimpse to details of future calculations with the representation Eq.(5) or Eq.(7), we take a look at the solution given by Eq.(48). This solution will be discussed later, but we glance over the borders of intervals for corresponding functions f and R there. They are depicted in Fig.1. Performing appropriate calculations, one should take into account that all variables and mappings are valid only in defined domains. Typically, the region suitable for calculations of Bogoliubov coefficients or total radiated energies is covered by two subsequent maps Δ_n . It makes evaluations of integrations and derivations of formulas more complex. This is the price we have paid for the replacement of Moore's equation by simpler relation Eq.(8). However, we will convince ourselves that this way is effective as a method for obtaining analytical results. Later, the details of calculations will be always skipped, so that the pattern in Fig.1 is the only commentary on practical calculations.

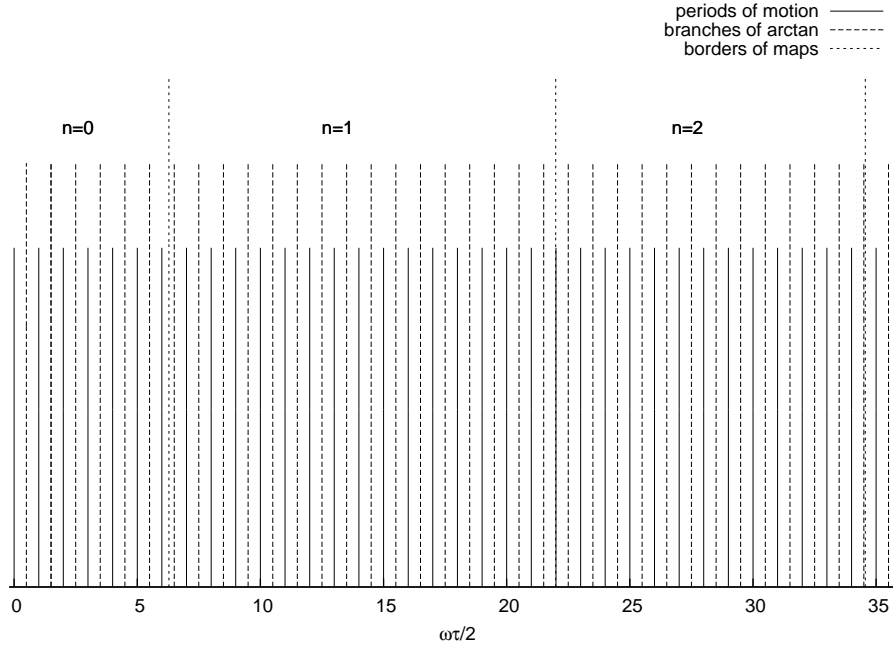


Figure 1: The borders of different change intervals for the Moore's function R defined by Eq.(5) with Eq.(48).

It is difficult to derive the function $R(\tau)$ from Eq.(2) for some prescribed trajectory $L(t)$. A great number of numerical approaches and approximate solutions were presented in other papers, but only few exact solutions are known. One way to obtain exact solutions is to specify the function $f(\tau)$, and then the trajectory $L(t)$ can be given in a parametric form:

$$\begin{cases} t &= [\tau + f(\tau)]/2 \\ L(t) &= [\tau - f(\tau)]/2 \end{cases} \quad (12)$$

The prescribed function $f(\tau)$ represents an admissible physical trajectory provide that it fulfils the following requirements [48]:

$$\begin{aligned} (i) \quad & f(\tau) = \tau - 2L \quad \text{for } \tau < L \\ (ii) \quad & \frac{1-v_{max}}{1+v_{max}} \leq \dot{f}(\tau) \leq \frac{1+v_{max}}{1-v_{max}} \\ (iii) \quad & f(\tau) < \tau \end{aligned} \quad (13)$$

In this paper, we will be exploiting the representation Eq.(5) to describe solutions of equations for the electromagnetic field in an oscillating one-dimensional cavity. Before we start with the construction of solutions, we describe general properties of fundamental maps Δ_1 extracted from proper solutions.

III. GENERAL PROPERTIES OF FUNDAMENTAL MAPS Δ_1

Knowledge of Moore's function enables us to draw out all information about the vibrating cavity system. The most important object to calculate is the vacuum expectation value of the energy density:

$$\langle T_{00}(t, x) \rangle = \varrho(t+x) + \varrho(t-x) . \quad (14)$$

Using appropriate formulas given in [64] and our representation Eq.(5), we can easily calculate:

$$\varrho(\tau) = -\frac{\omega^2}{48\pi} + \frac{\omega^2 - \omega_1^2}{48\pi} \left[\frac{1+v^2}{1+\Delta_{n(\tau)}^2(v)} \Delta'_{n(\tau)}(v) \right]^2 - \frac{\omega^2}{96\pi} (1+v^2)^2 S[\Delta_{n(\tau)}](v) , \quad (15)$$

where $v = \tan(\omega\tau/2)$ and $S[\Delta_{n(\tau)}](v)$ denotes the Schwartz derivative of $\Delta_{n(\tau)}$ with respect to v . The total quantum energy radiated from the cavity can be calculated from:

$$E(t) = \int_0^{L(t)} dx \langle T_{00}(t, x) \rangle = \int_{t-L(t)}^{t+L(t)} d\tau \varrho(\tau) = \frac{2}{\omega} \int \frac{dv}{1+v^2} \varrho(v) . \quad (16)$$

The most useful is the last formula which enables us to calculate the total energy by integration with respect to v . However, we should remember from the comment in the previous section on the pattern in Fig.1 that the replacement $v = \tan(\omega\tau/2)$ is valid only for a single period of cavity motion. The interval of integration $[t-L(t), t+L(t)]$ is to be divided into parts representing separate periods of motion. The map number $n(\tau)$ may change at most once per period.

Let us remind the relation $\Delta_n = (\Delta_1)^{\circ n}$, and we need only to specify the fundamental map Δ_1 . The knowledge of this map makes it possible to predict the evolution of the system and describe the resonance behavior. Henceforth, our exploration of a quantum field theory system is quite similar to examination of classical mechanics models under the parametric resonance [73]. We need only to analyze the asymptotic behavior of iterations of the mapping ruled by Δ_1 , which is known from the first period of motion.

We are going to make a list of general properties of fundamental maps Δ_1 . First, we include information that the cavity is assumed to be static in the past, i.e. for times $t < 0$. It follows that a solution for $t > 0$ is to be sewed together with a static one at $t = 0$. In our context, there is no need to demand that the sewing is perfectly smooth. For instance, we can accept that a force which causes cavity motion may be suddenly switched on. Such a solution may lead to some Dirac delta terms in its function for energy density, but from physical point of view the solution is acceptable and useful for applications, so that it is definitely worth saving them. Henceforth, let us propose some minimal set of requirements for sewing. We put forward three sewing conditions at the initial time $t = 0$. The trajectory of the cavity wall and its velocity should be continuous: $L(t=0) = L$ and $\dot{L}(t=0) = 0$. Moreover, there should be no sudden local growth of energy: $\langle T_{00}(t=0, x) \rangle = -\pi/(24L^2)$, i.e. the local energy density matches the Casimir energy density of vacuum fluctuations at the initial time. It is now straightforward to gather a full set of initial conditions for the fundamental map $\Delta_1(v)$:

$$\begin{aligned} \Delta_1(v_0) &= -v_0 \\ \Delta'_1(v_0) &= 1 \\ S[\Delta_1](v_0) &= 0 \end{aligned} ; \quad v_0 \equiv \tan \frac{\omega L}{2} = \tan \left(\pi \frac{L}{T} \right) = \tan \left(\frac{\pi}{2} \frac{\omega}{\omega_1} \right) . \quad (17)$$

The last condition implies that the construction of a fundamental map is yet a non-linear problem. Next, we impose the requirement that the velocity of cavity wall should never exceed v_{max} . The maximal velocity is a parameter of the cavity model and the only limitation is that $v_{max} < 1$. From Eq.(13)(ii) we obtain:

$$\frac{1-v_{max}}{1+v_{max}} \leq \frac{1+v^2}{1+\Delta_1^2(v)} \Delta'_1(v) \leq \frac{1+v_{max}}{1-v_{max}} . \quad (18)$$

This is a strong constraint on possible maps. One immediate consequence is that our function is increasing: $\Delta'_1(v) > 0$. We can also learn about its singularities from Eq.(18). If the function $\Delta_1(v)$ is singular at some v_s :

$$\lim_{v \rightarrow v_s \mp 0} \Delta_1(v) = \pm \infty , \quad (19)$$

then it is easy to prove that the following limit is finite and different from zero:

$$\lim_{v \rightarrow v_s} \frac{\Delta'_1(v)}{\Delta_1^2(v)} = - \lim_{v \rightarrow v_s} \frac{1}{(v-v_s)\Delta_1(v)} . \quad (20)$$

Henceforth, it follows the function $\Delta_1(v)$ may have only poles of order one:

$$\Delta_1(v) = \frac{h(v)}{(v - v_1)(v - v_2)\dots(v - v_s)} , \quad (21)$$

where the numerator $h(v)$ is an analytical function. Taking Eq.(18) together with Eq.(21), we note that for large values of v the function $h(v)$ shows the following asymptotic:

$$h(v) \sim |v|^k , \quad k \in \{s - 1, s, s + 1\} . \quad (22)$$

Finally, we look at the representation Eq.(7) and the periodicity condition Eq.(9). We conclude that the number of singularities s in the map Δ_1 for the representation Eq.(7) is at most one. Actually, we could replace ω with $s\omega$ in the representation Eq.(7) and allow for more complex form defined by Eq.(21). The same performance as that in Section IV might give new exact closed form solutions, but we will not examine this idea here.

In general, for large arguments either the function Δ_1 is unbounded or it takes a finite limit. Therefore, the respective continuity condition corresponds to one of two choices:

$$\Delta_1(\pm\infty) = \pm\infty \quad \text{or} \quad \Delta_1(-\infty) = \Delta_1(+\infty) = \text{finite value} . \quad (23)$$

Let us summarize this section. The basic set of solutions Eq.(1) for a quantum cavity system can be fully specified by Moore's function Eq.(5). In turn, this function is to be reconstructed from the fundamental map Δ_1 . The fundamental map is associated with the first period of motion. Some basic physical requirements lead to strong mathematical conditions on the application of function Δ_1 to cavity models which are admissible from physical point of view. This includes suitable sewing conditions Eq.(17) at some distinguished point v_0 , the inequalities Eq.(18) introduced by a limitation v_{max} on a cavity wall velocity and continuity condition Eq.(23). Moreover, the function Δ_1 may have at most one singularity (only a simple pole) and it behaves for large arguments according to Eq.(22) (s is a number of singularities, i.e. 0 or 1 here).

For some given fundamental map Δ_1 that fulfils all required mathematical conditions, it may be still difficult to derive trajectory $L(t)$ from Eq.(7) and Eq.(12) or map ranges L_n and index function $n(\tau)$ from Eq.(11). However, it is possible for rational functions. In the following section, we will discuss such solutions. They form a big and interesting family of exactly solvable cavity models. In particular, they include all examples of exact closed form solutions on vibrating cavities known from other papers [21, 41].

IV. EXACT CLOSED FORM ANALYTICAL SOLUTIONS

We use the considerations of the previous sections to find a family of exactly solvable quantum models of vibrating cavities. The static cavity length L is fixed and it characterizes a physical scale. According to the naive understanding of parametric resonance, the frequency of vibrations ω should be close to one of the resonance frequencies ω_N . It means that L is close to $NT/2$. However, it should be confirmed in a specific cavity model whether this naive criterion of resonance is justified. Moreover, it turns out there is a more subtle situation when L is an odd multiplicity of $T/2$, i.e. the parameter v_0 in Eq.(17) is infinite. Such cases should be analyzed in our treatment separately.

A. Linear fundamental maps Δ_1

We begin by addressing the case when a fundamental map Δ_1 is a polynomial. The condition Eq.(18) that velocities are not approaching the speed of light is very restrictive here. It allows only for a linear function. First, we will examine the case when v_0 is finite.

1. Finite values of v_0

Our method of proceeding follows closely on the formalism presented in the previous sections. Inserting a linear function into conditions (17), we pick out:

$$\Delta_1(v) = v - 2v_0 ; \quad v \equiv \tan \frac{\omega\tau}{2} . \quad (24)$$

It is easy to verify that the above function fulfils all physical requirements Eq.(17), Eq.(18) and Eq.(23). Let us define the natural number M and the angle parameter θ by:

$$L = (M + \frac{\theta}{\pi})T ; \quad M = 1, 2, 3, \dots , \quad |\theta| < \frac{\pi}{2} . \quad (25)$$

The parameter M can be interpreted as the order of the resonance. We will go through this subsection and see that the parameter M is better to characterize the resonance channel than N . The auxiliary function f for $\tau \geq L$ from Eq.(7) and its inverse function f^{-1} for $\tau \geq -L$ yield:

$$\begin{aligned} f(\tau) &= \frac{2}{\omega} \arctan \left(\tan \frac{\omega\tau}{2} - 2v_0 \right) + \left(\lfloor \frac{\tau}{T} + \frac{1}{2} \rfloor - 2M \right) T , \\ f^{-1}(\tau) &= \frac{2}{\omega} \arctan \left(\tan \frac{\omega\tau}{2} + 2v_0 \right) + \left(\lfloor \frac{\tau}{T} + \frac{1}{2} \rfloor + 2M \right) T , \\ v_0 &= \tan \frac{\omega L}{2} = \tan \theta . \end{aligned} \quad (26)$$

The corresponding trajectory of the cavity wall for $t \geq 0$ is to be evaluated from Eq.(12). Using some trigonometric identities, we reveal the following path:

$$L(t) = L + \frac{1}{\omega} \arcsin (\sin \theta \cos (\omega t)) - \frac{\theta}{\omega} . \quad (27)$$

For small parameters θ , the oscillations of the cavity wall are close to a sinusoidal wave (see Fig.2). With increasing θ ,

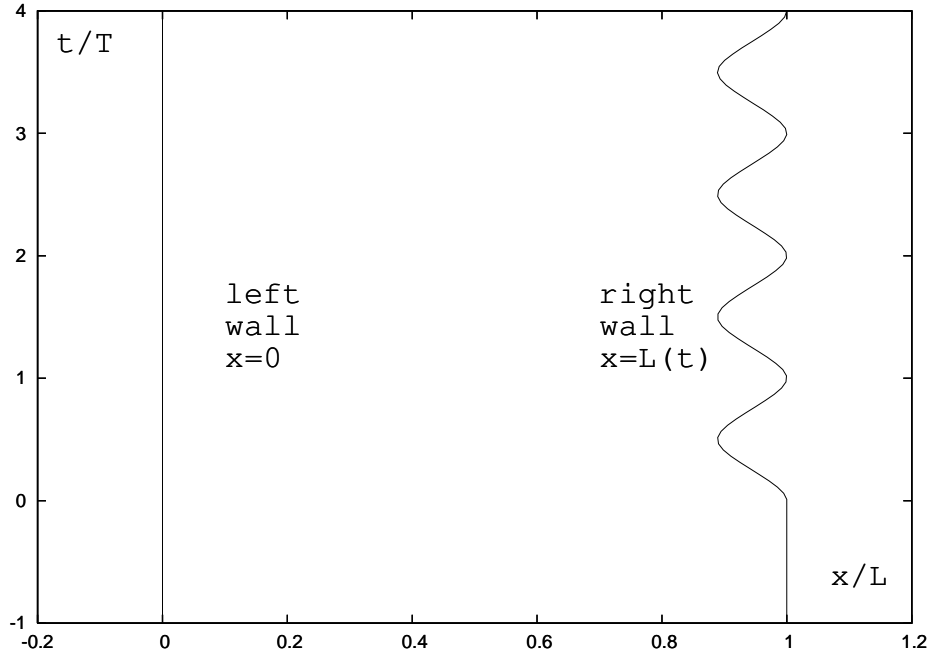


Figure 2: The trajectories of the cavity walls for the cavity motion Eq.(27) with $M = 2$ and $\theta = \pi/4$.

they are nearer to a triangle wave. The wall oscillations take place between MT and L . The amplitude of vibrations is $\Delta L = 2|\theta|/\omega$, and the maximal velocity yields:

$$v_{max} = |\sin \theta| . \quad (28)$$

The Moore's function for $\tau \geq L$ can be calculated from Eq.(10):

$$R(\tau) = \frac{2}{\omega} \arctan \left(\tan \frac{\omega\tau}{2} - 2n(\tau) \tan \theta \right) + \text{shift} . \quad (29)$$

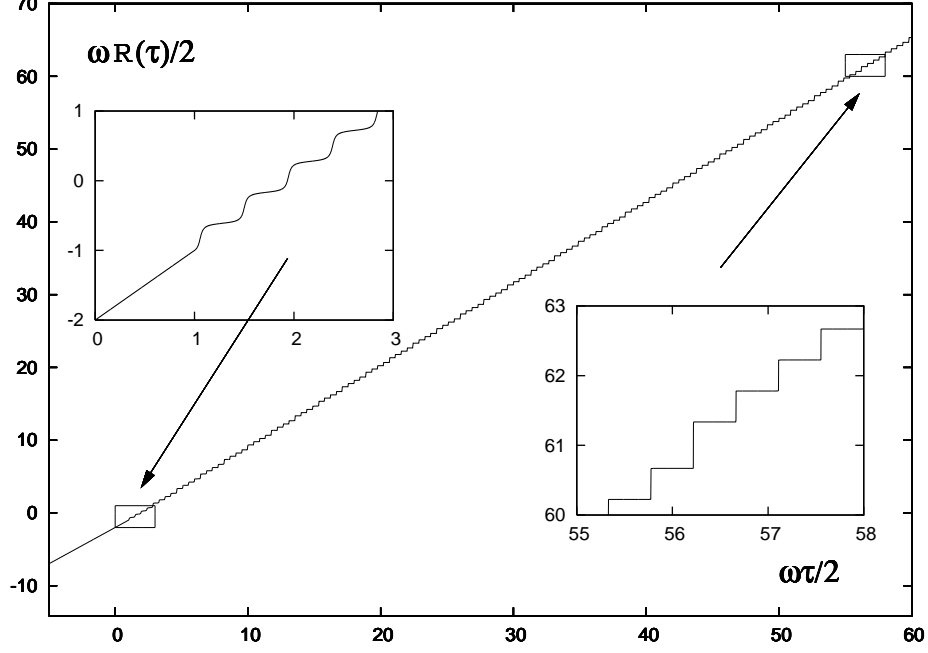


Figure 3: Moore's function for the cavity motion Eq.(27) with $M = 2$ and $\theta = \pi/4$.

The representation Eq.(29) can be effectively used if we are able to assign appropriate maps. It is nice that we are in a position to calculate the milestones L_n and the map number $n(\tau)$ from Eq.(11) exactly:

$$\begin{aligned}
 L_n &= \frac{2}{\omega} \arctan(2n+1)v_0 + (2n+1)MT, \\
 n(\tau) &= n_0(\tau) - 1 + \Theta(\tau - L_{n_0(\tau)-1}) + \Theta(\tau - L_{n_0(\tau)}), \\
 n_0(\tau) &= \lfloor \tau/(2MT) + 1/2 \rfloor,
 \end{aligned} \tag{30}$$

where the Heaviside step function is defined with $\Theta(0) = 1$. The Moore's function $R(\tau)$ from Eq.(29) for some specific motion of type Eq.(27) is shown in Fig.3. This function is always a small deviation from the linear function Eq.(6) that describes the static case. Disturbances caused by the cavity motion are magnified in Fig.3 for small and large function arguments. With increasing arguments they approach a well-known staircase shape ("devil's staircase"). However, the steps are hardly regular. If we look at the asymptotic behavior of the Moore's function, then we are convinced that it is not a good object for practical calculations, both analytical (perturbation methods) and numerical. Thus, the transformations for phase functions like Eq.(5) are necessary to get a feasible way to perform mathematical analysis of vibrating cavities in quantum field theory.

The shape function for the energy density Eq.(15) for the solution Eq.(24) reads:

$$\rho(\tau) = -\frac{\omega^2}{48\pi} + \frac{\omega^2 - \omega_1^2}{48\pi} \left[\frac{1 + v^2}{1 + (v - 2n(\tau) \tan \theta)^2} \right]^2. \tag{31}$$

A snapshot of the energy density is displayed in Fig.4. In general, there are M wave packets travelling left and M wave packets travelling right. Their localization and their evolution can be easily derived and it is in full agreement with results of procedures described in [63] and generalized in [68]. One can successfully derive periodic optical paths and calculate cumulative Doppler factors, cumulative conformal anomaly contributions and other quantities. Here, we skip such details. Far from the narrow packets, in the so-called sub-Casimir region [21] the energy density is constant and its asymptotic value is:

$$T_{00}^{out}(\tau) \cong -\frac{\omega^2}{24\pi} = -\left(2M + \frac{2\theta}{\pi}\right)^2 \rho_0, \quad \text{for large } \tau, \tag{32}$$

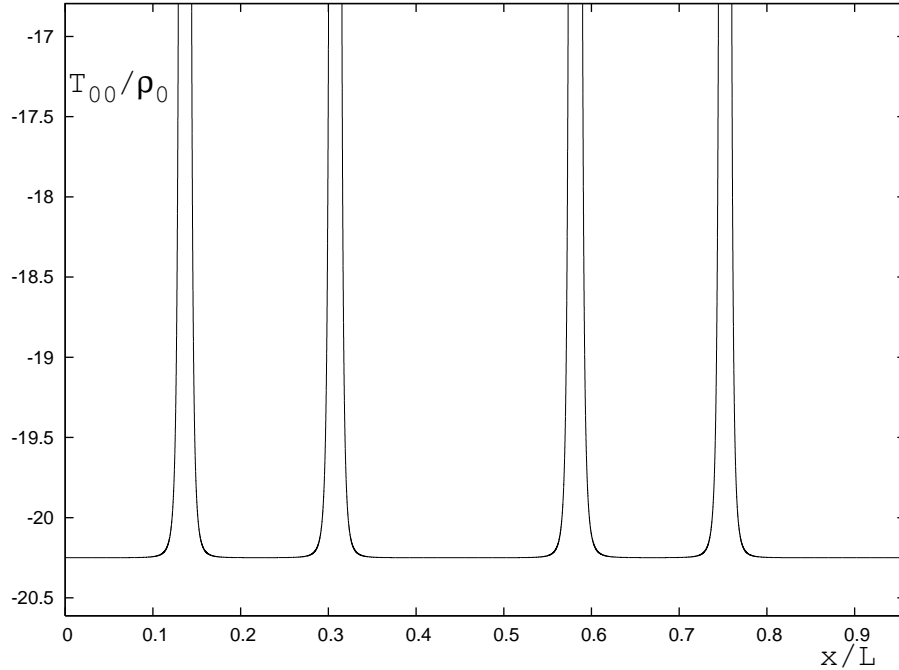


Figure 4: The energy density of the cavity Eq.(27) with $M = 2$ and $\theta = \pi/4$ defined in terms of the Casimir energy density of the static cavity $\rho_0 = \pi/(24L^2)$.

where $\rho_0 = \pi/(24L^2)$ is the magnitude of Casimir energy density for a static cavity of length L . Most of the energy is concentrated in narrow wave packets. The heights of peaks are proportional to t^4 , and their widths shrink like t^{-2} . It suggests that the total energy grows with time like t^2 . It is true, and one can calculate from Eq.(16) an exact formula. Here, we give only an asymptotic formula for large times:

$$E(t) \cong \frac{\omega(\omega^2 - \omega_1^2)}{24M\pi} (\tan \theta)^2 t^2, \quad t \gg 1. \quad (33)$$

As usual, there is no resonant behavior for the lowest resonance frequency. However, the resonance emerges for all frequencies above this threshold: $\omega > \omega_1$ (or equivalently: for $L > T/2$). Paradoxically, the resonance appears here for all frequencies but resonance ones. For resonance frequencies ω_N , either the cavity is static or the motion is singular (a triangular wave trajectory). Therefore, we should learn that the resonance frequencies are auxiliary objects, and real behavior of any physical cavity system depends on its individual features. A specific feature of the solution Eq.(27) is that for some fixed initial cavity length L , the resonance appears for almost all frequencies above some threshold. However, there is no possibility to adjust the amplitude of vibrations. There exists an exact relation between the amplitude and the frequency:

$$\frac{\Delta L}{L} = \left| 1 - \frac{2\omega_1}{\omega} \left\lfloor \frac{\omega}{2\omega_1} + \frac{1}{2} \right\rfloor \right|. \quad (34)$$

The most important problem for any linear dynamical system that exhibits a parametric resonance phenomenon is to find stable and unstable regimes for periodically excited parameters. Usually, the parametric resonance domains are depending on three crucial parameters: frequency and amplitude of periodic excitation and damping coefficient. The relevant Fig.5 exhibits the phase diagram for the cavity model described in this section. Since the frequency and the amplitude are related by Eq.(34), the unstable solutions Eq.(27) are represented by points on the curve to this plot. The instability of solutions is quadratic according to Eq.(33). It was suggested in [63] that a cavity model with a power-like instability appears as some boundary limit. If we extended the model Eq.(27) to possess more free parameters, then other points in Fig.5 would represent cavity motions when the amplitude and the frequency do not match Eq.(34). Below the border curve, the cavity system would be stable, and for states represented by points that are placed above the curve we would observe the resonance with the exponential growth of the total radiated energy. The last example of solution considered in this paper will justify such predictions. However, we are not able to prove

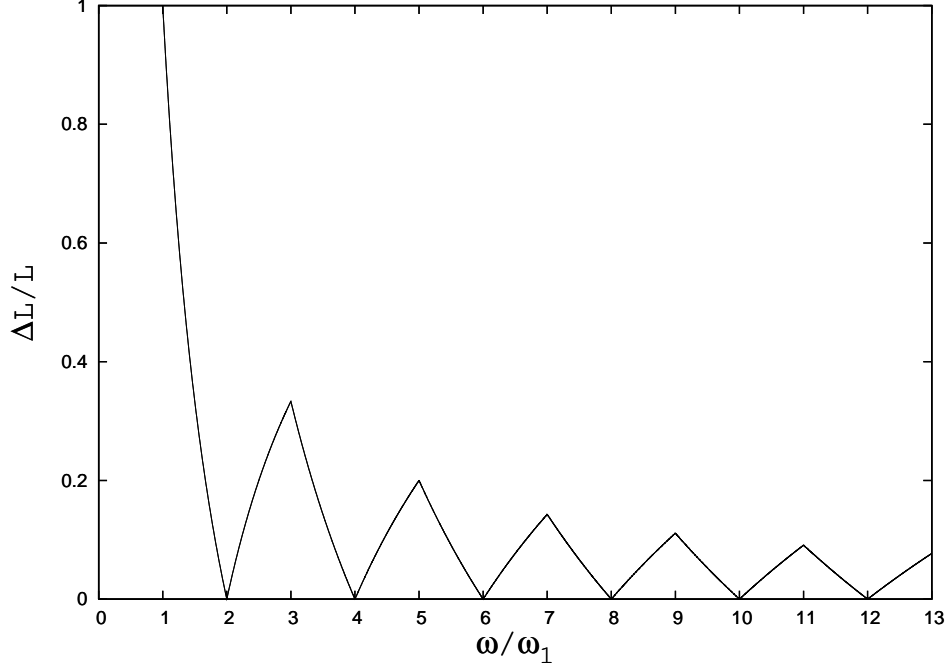


Figure 5: The phase diagram for the cavity model Eq.(27): the relative amplitude of cavity oscillations versus the frequency as the multiplicity of the fundamental frequency.

that the statement is generally true. Note famous Arnold's tongue structure [73] in Fig.5. However, the tongues are rather broad. In the classical theories [73], Arnold's tongue has usually a narrow knife shape.

2. Infinite values of v_0

We get infinite values of sewing points in Eq.(17) if the cavity oscillates at odd resonance frequencies $\omega = \omega_{2M-1}$,

$$L = (M - \frac{1}{2})T ; \quad M = 1, 2, 3, \dots \quad (35)$$

The conditions Eq.(17), Eq.(18) and Eq.(23) are satisfied by a linear map with an arbitrary intercept parameterized by θ (warning: θ has different meaning than the same parameter in the previous subsection. For convenience, we have redefined this parameter here in such a way that numerous formulae match those of the previous section):

$$\Delta_1(v) = v - 2 \tan \theta ; \quad |\theta| < \frac{\pi}{2} \quad (36)$$

The trajectory of the cavity wall $L(t)$ is the same as in Eq.(27). But now, the physical situation is different. In the previous section, we were almost free to adjust the frequency of the oscillations. If the frequency were fixed, the amplitude would be given by Eq.(34). Here, the frequency is not arbitrary, but the amplitude of the oscillations $2|\theta|/\omega$ is adjustable. These solutions are already known and they were presented first in [41] (they correspond to the solutions numbered by $m \equiv 2N - 1$ using the notation of that paper). Our auxiliary functions are slightly modified:

$$\begin{aligned} f(\tau) &= \frac{2}{\omega} \arctan(\tan \frac{\omega\tau}{2} + 2 \tan \theta) + (\lfloor \frac{\tau}{T} + \frac{1}{2} \rfloor - 2M + 1) T , \\ f^{-1}(\tau) &= \frac{2}{\omega} \arctan(\tan \frac{\omega\tau}{2} - 2 \tan \theta) + (\lfloor \frac{\tau}{T} + \frac{1}{2} \rfloor + 2M - 1) T . \end{aligned} \quad (37)$$

Again $v_{max} = |\sin \theta|$. The milestones L_n and the map number $n(\tau)$ are given by much simpler formulae:

$$\begin{aligned} L_n &= (2n + 1)L, \\ n(\tau) &= \lfloor \frac{\tau}{2L} + \frac{1}{2} \rfloor. \end{aligned} \quad (38)$$

The Moore's function is given by the following formula:

$$R(\tau) = \frac{2}{\omega} \arctan \left(\tan \frac{\omega\tau}{2} - 2n(\tau) \tan \theta \right) + \lfloor \frac{\tau}{T} + \frac{1}{2} \rfloor T - 2L. \quad (39)$$

The profile function for the energy density is given by:

$$\rho(\tau) = -\frac{(2M-1)^2\pi}{48L^2} + \frac{M(M-1)\pi}{12L^2} \left[\frac{1+v^2}{1+(v+2n(\tau)\tan\theta)^2} \right]^2. \quad (40)$$

Now, it is much more easier to work out the integral Eq.(16). Actually, it is interesting to look into an exact and closed form formula for the total energy produced inside the cavity:

$$\begin{aligned} E(t) = & \frac{M(M-1)\pi \tan^2 \theta}{12L^3} t^2 + \\ & + \frac{M(M-1)\tan^2 \theta}{3(2M-1)L^2} \left[\frac{\pi}{2} \text{sign } \theta - \arctan \left(\frac{1}{\tan(\omega t/2)} \frac{1+\sqrt{1+\sin^2 \omega t \tan^2 \theta} - (t/L+1-2\alpha(t)) \sin \omega t \tan \theta}{1-\sqrt{1+\sin^2 \omega t \tan^2 \theta} + (t/L+1-2\alpha(t)) \sin \omega t \tan \theta} \right) \right] (t + (1-2\alpha(t))L) + \\ & - \frac{(2M-1)^2\pi}{24L^2} L(t) + \frac{M(M-1)\pi}{6L} + \frac{M(M-1)\pi \tan^2 \theta}{3L} \alpha(t)(1-\alpha(t)) + \\ & \frac{M(M-1)\tan \theta}{3(2M-1)L} \frac{1+2(t/(2L)-\alpha(t))^2 \tan^2 \theta + (t/L+1-2\alpha(t)) \tan(\omega(t+L(t))/2) \tan \theta}{1+(\tan(\omega(t+L(t))/2) + (t/L+2-2\alpha(t)) \tan \theta)^2}, \end{aligned} \quad (41)$$

where

$$\alpha(t) \equiv \frac{t}{2L} - \lfloor \frac{t}{2L} \rfloor. \quad (42)$$

Similarly to the solution presented in the previous subsection, the total energy of the system grows quadratically with time. We have extracted the leading term. However, the next to leading terms that are linear in time play an important role as well. They cause that the energy is not irradiated continuously but rather in sudden jumps. To verify that, we should take two leading terms from Eq.(41) and make the approximation for large values of t . As a result we obtain:

$$E(t) \cong \frac{M(M-1)\pi \tan^2 \theta}{3(2M-1)^2 L} \left(\lfloor \frac{t}{T} \rfloor + \Theta(\theta) \right)^2 \quad (43)$$

The presence of Heaviside function means that the problem is not analytical in the parameter θ , i.e. with respect to the change of direction of oscillations. We see that impulses of energy growth occur every period. For small amplitudes, the energy is proportional to the square of the amplitude. This is in a agreement with a non-relativistic limit of small velocities. It is amusing to consider a quasi-classical analogue of the model. Suppose, that at the initial state we have an uniform distribution of energy of classical fields. The value of the energy density equals the absolute value of the static Casimir energy: $\rho_0 = \pi/24L^2$. It corresponds to the classical potential $A(t, x) = \varphi(t+x) + \varphi(t-x)$ with $\varphi(\tau) = \pi\tau/48L^2$. The classical energy is given by [63]:

$$E_{cl}(t) = \int_0^{L(t)} dx T_{00}(t, x) = \int_{t-L(t)}^{t+L(t)} d\tau \dot{\varphi}^2(\tau). \quad (44)$$

Next, we allow for the classical evolution of the electromagnetic system. From classical equations of motion we get: $\varphi(\tau) = \varphi(f(\tau))$. Using initial conditions, we obtain a classical global solution:

$$\varphi(\tau) = \frac{\pi}{48L^2} R(\tau). \quad (45)$$

We have encountered almost the same asymptotic formula for the energy as in the quantum case. The only exception is the coefficient:

$$E_{cl}(t) \cong \frac{\pi \tan^2 \theta}{12(2M-1)^2 L} \left(\lfloor \frac{t}{T} \rfloor + \Theta(\theta) \right)^2 \quad (46)$$

The results are confronted in Fig.6. We plot coefficients of asymptotic energy formulae for the first eight resonance channels. To make plots more readable, we have used continuous lines. Classically, the strongest growth is for the fundamental resonance frequency, next resonances are less effective. For the quantum case, the situation reverses. Due to the effect of quantum anomaly, there is no resonance in the first channel. Next, there is a rapid saturation for higher resonance channels. In units of $\pi \tan^2 \theta / 12L$, the sum of a classical coefficient and a quantum coefficient is just a unity.

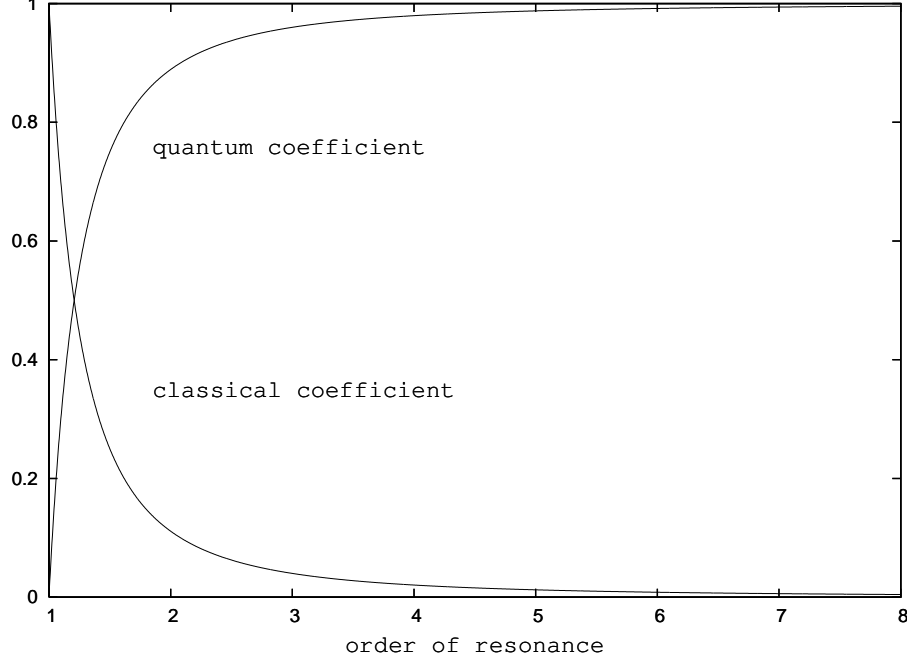


Figure 6: Coefficients for quantum Eq.(43) and classical Eq.(46) asymptotic energy formulae. The coefficients are in units of $\pi \tan^2 \theta / 12L$.

B. Homographic fundamental maps Δ_1

We now turn the discussion to the case of maps Δ_1 that are rational functions with single poles:

$$\Delta_1(v) = \frac{h(v)}{v - v_0} . \quad (47)$$

From Eq.(18) we find that $h(v)$ is at most a quadratic function. The periodicity condition Eq.(9) allows only for one singularity of Δ_1 per period. Therefore, we can only consider homographic maps. It is convenient for us to start the discussion with inversions, and then we will be looking at a general case.

1. Inversion map

We evaluate that Eq.(17) and other necessary conditions are satisfied by maps:

$$\Delta_1(v) = -\frac{v_0^2}{v} , \quad v_0 = \tan \frac{\omega L}{2} = \tan \theta . \quad (48)$$

There are no solutions for singular v_0 , so that $\omega \neq \omega_{2N-1}$ and $|\theta| < \pi/2$. Moreover, we are forced to assume $\theta \neq 0$ and this way all resonance frequencies are excluded here: $\omega \neq \omega_N$. The auxiliary function f for $\tau \geq L$ from Eq.(7) and its inverse function f^{-1} for $\tau \geq -L$ yield:

$$\begin{aligned} f(\tau) &= \frac{2}{\omega} \arctan(v_0^2 / \tan \frac{\omega \tau}{2}) + (\lfloor \frac{\tau}{T} \rfloor - 2M + \Theta(-\theta)) T , \\ f^{-1}(\tau) &= \frac{2}{\omega} \arctan(v_0^2 / \tan \frac{\omega \tau}{2}) + (\lfloor \frac{\tau}{T} \rfloor + 2M + \Theta(\theta)) T , \\ L &= (M + \frac{\theta}{\pi}) T . \end{aligned} \quad (49)$$

The trajectory of the cavity wall for $t > 0$ is reconstructed from Eq.(12):

$$L(t) = L - \frac{2\theta}{\omega} + \frac{\text{sign } \theta}{\omega} \left[\frac{\pi}{2} - \arcsin(\cos 2\theta \cos(\omega t)) \right] . \quad (50)$$

Evidently, the maximal velocity is now:

$$v_{max} = |\cos 2\theta| . \quad (51)$$

In the limit $\omega \rightarrow \omega_N$, we encounter a triangle wave trajectory. For $\omega = (\omega_N + \omega_{N-1})/2$, our solution degenerates to a static one. The oscillations do always take place between L and $L + \text{sign}\theta(\pi - 4|\theta|)/\omega$. The corresponding milestones for our representation of Moore's function are given by:

$$L_n = (-1)^n L + \lfloor \frac{n+1}{2} \rfloor (4M + \text{sign}\theta)T . \quad (52)$$

We make the energy density explicit:

$$\rho(\tau) = \begin{cases} -\frac{\omega_1^2}{48\pi} & \text{for } \tau \in [L_{2k-1}, L_{2k}) \\ -\frac{\omega^2}{48\pi} + \frac{\omega^2 - \omega_1^2}{48\pi} \frac{v_0^2(1+v^2)}{v_0^4 + v^2} & \text{for } \tau \in [L_{2k}, L_{2k+1}) \end{cases} \quad (53)$$

There are wave packets in the energy density, but there is no unbounded growth of the total energy. The quantum cavity system is stable and its total accumulated energy oscillates with the period $(4M + \text{sign}\theta)T$.

The solution is also well-defined for $\omega < \omega_1$ ($L < T/2$). It corresponds to $M = 0$ and $\theta > 0$ in Eq.(49) and Eq.(50). Here, as the only effect of the cavity motion there are pits in the energy density (negative wave packets) that may appear periodically in synchronization with cavity oscillations.

2. Homographic map

As a final and the most interesting application of our ideas, we consider a solution with a fundamental map being a homographic function. So then, upon confrontation with initial conditions Eq.(17), we set:

$$\Delta_1(v) = -\frac{v_1 v + v_0(v_0 - 2v_1)}{v - v_1} , \quad (54)$$

where $v_0 = \tan(\omega L/2)$ and v_1 is an arbitrary parameter. It is straightforward to check that physical solutions exist on condition that $v_0 \neq v_1$. In passing, we note that the solutions that have been described in the previous subsection are reproduced for $v_1 = 0$.

The evaluation of relevant auxiliary functions f and f^{-1} ends with the results:

$$\begin{aligned} f(\tau) &= \frac{2}{\omega} \arctan \Delta_1(v) + \left(\lfloor \frac{\tau}{T} + \frac{1}{2} \rfloor - 2M + \Theta(v - v_1) - \Theta(v_0 - v_1) \right) T , \\ f^{-1}(\tau) &= \frac{2}{\omega} \arctan \Delta_1^{-1}(v) + \left(\lfloor \frac{\tau}{T} + \frac{1}{2} \rfloor + 2M + \Theta(v + v_1) - \Theta(v_1 - v_0) \right) T , \\ L &= (M + \frac{1}{\pi} \arctan v_0)T . \end{aligned} \quad (55)$$

The milestones are given by:

$$L_n = \frac{2}{\omega} \arctan \Delta_n^{-1}(v_0) + \left[(2n+1)M + \sum_{k=0}^{n-1} \Theta(\Delta_k^{-1}(v_0) + v_1) - n\Theta(v_1 - v_0) \right] T . \quad (56)$$

The angle parameter θ may be introduced here by using the following formula:

$$\tan \theta = \frac{1 + v_0^2}{2v_1} - v_0 . \quad (57)$$

With the above definition, the derivation of the trajectory of the cavity wall from Eq.(12) gives us:

$$\sin(\omega L(t) + \theta) = \sin(\omega L + \theta) \cos \omega t , \quad (58)$$

and it can be disentangled successfully:

$$L(t) = L + \frac{1}{\omega} [\arcsin(\sin(\omega L + \theta) \cos(\omega t)) - \arcsin(\sin(\omega L + \theta))] . \quad (59)$$

We have assumed throughout this paper that the functions \arcsin and \arctan have their ranges restricted to $[-\pi/2, \pi/2]$. It makes the right hand side of Eq.(59) uniquely and properly defined. The maximal velocity is:

$$v_{max} = |\sin(\omega L + \theta)|, \quad (60)$$

while the amplitude of oscillations is given by $\Delta L = (2/\omega) \arctan v_{max}$.

It is important to point some special cases of Eq.(59). For $M = 1$, $v_0 = 0$ and $v_1 = 1/(2 \tan \theta)$, we get a cavity model investigated by Law in [21]. It was the first exact closed form solution presented in literature. The generalization of this solution for any M is leading to a second set of solutions described in the paper by Wu et al. [41] (the solutions with $m = 2N$ in their notation). All solutions correspond to cavity vibrations with resonance frequencies $\omega = \omega_N$. It was established for Law's and Wu's solutions that there appears resonant instability with a power-like behavior. We are not going to discuss these solutions here and send the reader back to the original papers. We wish only to note that there is one more class of solutions with a power-like instability. We obtain these solutions if we put $v_0 = 2v_1$ in Eq.(54). Let us describe them very briefly. The frequency of oscillation is a free parameter and may be tuned to any value greater than the fundamental frequency ω_1 . But the amplitude of oscillations is uniquely defined by the choice of frequency.

The solutions with exponential instability are obtained for $v_0 \neq 0$ and $v_0 \neq 2v_1$. The derivation of the maps Δ_n for the representation of Moore's function Eq.(5) requires the calculation of n -fold composition of the fundamental map given by Eq.(54). It is easy for homographies, so that the result is:

$$\Delta_n(v) = \Delta_1^{\circ n}(v) = (\lambda_1 - \lambda_2) \frac{2(\lambda_1^n + \lambda_2^n)v + (\lambda_1 - \lambda_2)(\lambda_1^n - \lambda_2^n)}{4(\lambda_1^n - \lambda_2^n)v + 2(\lambda_1 - \lambda_2)(\lambda_1^n + \lambda_2^n)}, \quad (61)$$

where:

$$\lambda_{1,2} = -v_1 \pm \sqrt{v_0(2v_1 - v_0)}. \quad (62)$$

The profile function for the energy density is then given by:

$$\rho(\tau) = -\frac{\omega^2}{48\pi} + \frac{\omega^2 - \omega_1^2}{48\pi} (v_0 - v_1)^{2n(\tau)} \left[\frac{1 + v^2}{(\frac{\lambda_1^n + \lambda_2^n}{2}v + \frac{(\lambda_1 - \lambda_2)(\lambda_1^n - \lambda_2^n)}{4})^2 + (\frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}v + \frac{\lambda_1^n + \lambda_2^n}{2})^2} \right]^2. \quad (63)$$

We restrict ourselves to calculate only the approximate value of the total energy for large times. Therefore, the first term in Eq.(63) can be omitted, while the second term integrated over one period gives:

$$\int_{v=-\infty}^{v=+\infty} d\tau \rho(\tau) = \frac{\omega^2 - \omega_1^2}{48\pi} \frac{Tr(H^T H)}{det(H)}, \quad (64)$$

where H is a matrix composed of coefficients of the homography Δ_n in Eq.(61). It can be easily calculated that:

$$\frac{Tr(H^T H)}{det(H)} = \frac{1}{4} \left(v_1 + \frac{1}{v_1} \right)^2 \left[\left(\frac{\lambda_1}{\lambda_2} \right)^{n(\tau)} + \left(\frac{\lambda_2}{\lambda_1} \right)^{n(\tau)} \right] - \frac{1}{2} \left(v_1 - \frac{1}{v_1} \right)^2. \quad (65)$$

For simplicity, we have assumed here that the number $n(\tau)$ do not change in the interval of integration. Further, we assume small amplitudes: $2L(t) \approx 2L \approx 2MT$ and obtain the approximate formula:

$$E(t) \cong \frac{4M^2 - 1}{96} \left(v_1 + \frac{1}{v_1} \right)^2 \cosh \left[\frac{v_1 - \sqrt{v_0(2v_1 - v_0)}}{v_1 + \sqrt{v_0(2v_1 - v_0)}} \frac{t}{2L} \right]. \quad (66)$$

In the above brief calculation, we have demonstrated that it is rather easy and safely in our treatment to perform approximate calculations and skip insignificant details. In fact, the treatment described in Section II and III is well adapted for perturbative methods. However, the relation for the minimal amplitude of oscillations ΔL_{min} enough to trigger exponential instability of the cavity system vibrating at some fixed frequency ω can be derived exactly:

$$\frac{\Delta L_{min}}{L} = \frac{\left| \frac{\omega}{\omega_1} - \lfloor \frac{\omega}{\omega_1} + \frac{1}{2} \rfloor \right|}{\frac{\omega}{\omega_1}}. \quad (67)$$

As the velocity of cavity wall cannot reach a speed of light, we obtain also the upper limit for amplitudes of oscillations

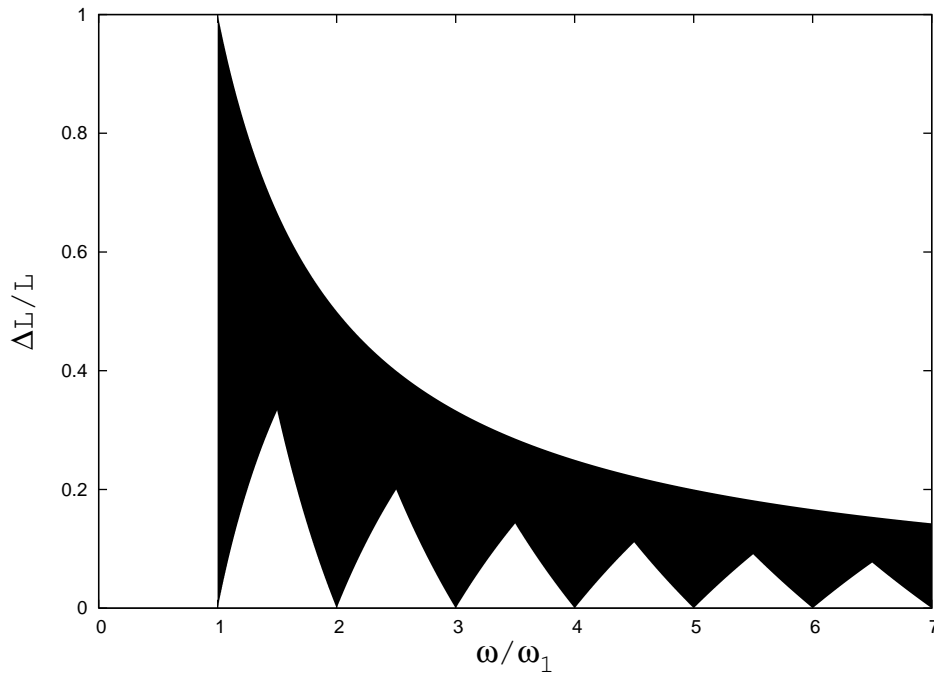


Figure 7: The phase diagram for stability and instability regions for the cavity model Eq.(50).

ΔL_{min} at given frequency ω :

$$\frac{\Delta L_{max}}{L} = \frac{\omega_1}{\omega} . \quad (68)$$

The above relation allows us to set up the phase diagram of stability and instability regions. A black area in Fig.7 correspond to the instability region of the vibrating cavity model Eq.(50). Below this area, we have defined frequencies and amplitudes the cavity model is stable. A marginal behavior appears when the energy grows quadratically with time. It is observed for $v_0 = 0$ (resonance frequencies, boundaries between adjacent Arnold's tongues) and $v_0 = 2v_1$ (boundary points between stability and instability regions). Other parts of the diagram correspond to points where the cavity model is not well defined (physical assumptions about $L(t)$ at the beginning of Section II are violated).

V. CONCLUSIONS

We have presented a rich class of exact and closed form analytical solutions for the quantum vacuum field in a one-dimensional cavity vibrating under the parametric resonance conditions. The solutions are valid for all times, frequencies of cavity oscillations and/or their amplitudes are free parameters. For small amplitudes, cavity oscillations are close to sinusoidal ones. In view of these properties, we can expect our solutions to yield all generic features known from other investigations on vibrating cavity models in a single dimension.

The representation of solutions Eq.(5) that appears in our treatment is based on $SL(2, R)$ symmetry of scalar fields quantized in a static cavity. We have introduced the notion of fundamental maps that are more convenient to proceed than Moore's phase functions. There is a direct mathematical relationship between iterations of fundamental maps and the mechanism of parametric resonance. This is the way we can get insight into the regions of stability and instability of the model (see Fig.7). One can calculate the rate of increase of the energy and the Lyapunov exponents. The stability-instability transition points and points between adjacent Arnold's tongues correspond to cavity models with a power-like instability. Thus, the most crucial questions can be tackled. If we insist on detailed calculations or exact formulas, then we have to set up how regions in space are covered by our maps (see Fig.1). Summarizing technical aspects, we can tackle with any solution successfully and completely provided that we know its fundamental map Δ_1 Eq.(7) and respective ranges of maps L_n together with $n(\tau)$ Eq.(11). In this paper, general properties of fundamental maps for any physically reasonable solutions have been described. This setup is also a good start point for perturbative calculations.

To the best of our knowledge, both exact closed form solutions for off resonant frequencies and exact closed form solutions with exponential instability for vibrating cavities were not presented before. It refers also to Arnold's phase diagrams of stability-instability regions for solutions of vibrating cavity models. It is also important to state that the mechanism of parametric resonance in the quantum field theory shares many common features with its analogue in the classical theory. We believe that this similarity could be maintained also for three-dimensional vibrating cavities. The same should be true for the relevance of the symmetry of the quantized cavity model.

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